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Linear recognition of generalized Fibonacci cubes $Q_h(111)$

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The generalized Fibonacci cube $Q_h(f)$ is the graph obtained from the $h$-cube $Q_h$ by removing all vertices that contain a given binary string $f$ as a substring. In particular, the vertex set of the 3rd order generalized Fibonacci cube $Q_h(111)$ is the set of all binary strings $b_1b_2\ldots b_h$ containing no three consecutive 1's. We present a new characterization of the 3rd order generalized Fibonacci cubes based on their recursive structure. The characterization is the basis for an algorithm which recognizes these graphs in linear time.

Keywords: generalized Fibonacci cube, 3rd order generalized Fibonacci cube, characterization, recognition algorithm

1 Introduction

The study of interconnection topologies is not just an important subject in the area of parallel or distributed systems, it also initiates the research work on several new interesting classes of graphs Liu and Hsu (1992). Besides hypercubes, Fibonacci cubes are the most known class of graphs which are applied as a model for interconnection network. They were studied from several points of view and they are known to possess many appealing properties Gregor (2006); Hsu (1993).

The Fibonacci cube $\Gamma_h$, $h \geq 1$, is defined as follows. The vertex set of $\Gamma_h$ is the set of all binary strings $b_1b_2\ldots b_h$ containing no two consecutive 1's. Two vertices are adjacent in $\Gamma_h$ if they differ in precisely one bit. Several structural properties and applications including different metric aspects such as recursive construction, hamiltonicity, degree sequence and other enumeration have been investigated Castro and Mollard (2012); Dedó et al. (2002); Klavžar (2005); Klavžar and Mollard (2012, 2014). For an extensive survey of Fibonacci cubes see Klavžar (2013).

Suppose $f$ is an arbitrary binary string and $h \geq 1$. The generalized Fibonacci cube, $Q_h(f)$, was introduced as the graph obtained from $Q_h$ by removing all vertices that contain $f$ as a substring Ilić et al. (2012). In this notation the Fibonacci cube $Q_h$ is $Q_h(11)$. The question for which strings $f$, $Q_h(f)$ is
Lemma 2.1 for Hsu and Chung (1993) If \( Q^F \) is a Fibonacci cube (of dimension \( h \)), then it is an isometric subgraph of \( Q_h \) raised and answered in [Hsu et al., 2012] for some small lengths of \( f \). Also, for a given large length of \( f \), the proportion of \( f \) such that \( Q_h(f) \) is an isometric subgraph of \( Q_h \) is investigated [Klavžar and Shpectorov, 2012].

The subclass of generalized Fibonacci cubes, the graphs \( Q_h(1^k) \), have been introduced already in [Hsu and Chung, 1993] and further studied in [Liu et al., 1993], [Wasserman and Ghozati, 2003], [Zagaglia Salvi, 1996]. They are called the \( k \)-th order "generalized Fibonacci cubes" in [Hsu and Chung, 1993].

Hsu and Chung raised the following question [Hsu and Chung, 1993]:

**Question 1.1** Given a graph, how to quickly decide whether it is the \( k \)-th order generalized Fibonacci cube of dimension \( h \)?

The question has been answered only for Fibonacci cubes in [Vesel, 2015] where a linear recognition algorithm for \( Q_h(1^3) \) is presented. This result provides a linear recognition algorithm for a closely related class of Lucas cubes [Taranenko, 2013].

The main contribution of this paper is an isometric dimension \( d \) of a graph \( G \) and \( d \) is defined as the isometric subgraph of hypercubes. The paper is organized as follows. In the next section we give basic definitions and concepts needed in this paper. In Section 3, a new characterization of the 3rd order generalized Fibonacci cube is given. This characterization is the basis for the algorithm presented in the last section, which recognizes these graphs in linear time.

## 2 Preliminaries

The hypercube of order \( h \), denoted by \( Q_h \), is the graph \( G = (V, E) \) where the vertex set \( V(G) \) is the set of all binary strings \( u = u_1u_2 \ldots u_h, u_i \in \{0, 1\} \), and two vertices \( x, y \in V(G) \) are adjacent in \( Q_h \) if and only if \( x \) and \( y \) differ in precisely one place.

Fibonacci numbers form a sequence of non-negative integers \( F_n \), where \( F_0 = 0, F_1 = 1 \) and for \( n \geq 0 \) satisfy the recurrence \( F_{n+2} = F_{n+1} + F_n \).

The \( k \)-th order generalized Fibonacci numbers form a sequence of positive integers \( F^k_n \), where \( F^k_0 = F^k_1 = \ldots = F^k_{k-2} = 0, F^k_{k-1} = 1 \) and for \( n \geq 0 \) satisfy the recurrence \( F^k_{n+k} = \sum_{i=0}^{k-1} F^k_{n+i} \).

We will use \( [n] \) for the set \( \{1, 2, \ldots, n\} \) in this paper.

The \( k \)-th order Fibonacci string of length \( h \) is a binary string \( u = u_1u_2 \ldots u_h, u_i \in \{0, 1\} \), with \( u_i \cdot u_{i+1} \ldots u_{i+k-1} = 0 \) for \( i \geq h-k+1 \). In other words, the \( k \)-th order Fibonacci string is a binary string without \( k \) consecutive ones.

The 3rd order generalized Fibonacci cube \( Q_h(111) \) is the subgraph of \( Q_h \) induced by the 3rd order Fibonacci strings of length \( h \). For convenience we also set \( Q_0(111) = K_1 \). The 3rd order generalized Fibonacci cubes \( Q_h(111) \) are shown in Fig. 1 for \( h = 1, 2, 3, 4 \). Note that \( Q_0(111) \) is isomorphic to \( Q_0 \) for \( h \leq 2 \), while \( Q_3(111) \) is isomorphic to the vertex-deleted subgraph of \( Q_3 \) denoted by \( Q_3^v \).

It is easy to see that the following lemma holds [see also Hsu and Chung, 1993].

**Lemma 2.1** If \( h \geq 0 \), then \( |V(Q_h(111))| = F^3_{h+3} \).

Let \( n \) denote the number of vertices of \( Q_h(111) \).

**Lemma 2.2** Hsu and Chung (1993) If \( h \geq 0 \), then \( |E(Q_h(111))| = \Theta(n \log n) \).

A subgraph \( H \) of a graph \( G \) is isometric if \( d_H(u, v) = d_G(u, v) \) for any pair of vertices \( u \) and \( v \) from \( H \). Isometric subgraphs of hypercubes are called partial cubes. The isometric dimension of a graph \( G \),
Let $\alpha : V(G) \rightarrow V(Q_h)$ be an isometric embedding. We will denote the $i$-th coordinate of $\alpha$ with $\alpha^i$, i.e. $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^h)$.

Let $G$ be a connected graph and $e = xy, f = uv$ be two edges of $G$. We say that $e$ is in relation $\Theta$ to $f$ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$. $\Theta$ is reflexive and symmetric, but need not be transitive. We denote its transitive closure by $\Theta^*$. It was proved in Winkler (1984) that $G$ is a partial cube if and only if $G$ is bipartite and $\Theta^* = \Theta$. The $\Theta$-classes of a partial cube $G$ constitute a partition of $E(G)$ and will be denoted with $1, \ldots, h$ in the sequel.

Let $G$ be a partial cube with $\text{idim}(G) = h$ and assume that we are given an isometric embedding of $G$ into $Q_h$. Each pair $(i, \chi) \in [h] \times \{0, 1\}$ defines the semicube $W_{(i, \chi)} = \{u \in V(G) | \alpha^i(u) = \chi\}$. For any $i \in [h]$, we call $W_{(i, 0)}, W_{(i, 1)}$ a complementary pair of semicubes.

Any isometric embedding of $G$ with $\text{idim}(G) = h$ describes the same family of semicubes and pairs of complementary semicubes, which are indexed in a different way. For a partial cube $G$ and a complementary pair of semicubes $W_{(i, 0)}, W_{(i, 1)}$, the set of edges with one end vertex in $W_{(i, 0)}$ and the other in $W_{(i, 1)}$ forms a $\Theta$-class $i$ of $G$.

For an edge $ab$ of $G$ we write:

$W_{ab} = \{w \in V(G) | d(a, w) < d(b, w)\}$,

$W_{ba} = \{w \in V(G) | d(b, w) < d(a, w)\}$,

$F_{ab} = \{xy | xy \text{ edge of } E(G) \text{ with } x \in W_{ab} \text{ and } y \in W_{ba}\}$,

$U_{ab} = \{w \in W_{ab} | w \text{ is an end vertex of an edge in } F_{ab}\}$,

$U_{ba} = \{w \in W_{ba} | w \text{ is an end vertex of an edge in } F_{ab}\}$.

We will need the following well known lemma, cf. [Hammack et al., 2011, Proposition 11.7].

**Lemma 2.3** Let $e = ab$ be an edge of a connected bipartite graph $G$ and $F_{ab} = \{f | f \in E(G), e\Theta f\}$. Then $G - F_{ab}$ has exactly two connected components, namely $G[W_{ab}]$ and $G[W_{ba}]$.

For an edge $ab$ of a partial cube with an isometric embedding $\alpha$ into $Q_h$, let the vertices $a$ and $b$ differ in the coordinate $i$, i.e. $\alpha^i(a) = 0$ and $\alpha^i(b) = 1$. Then $W_{ab} = W_{(i, 0)}, W_{ba} = W_{(i, 1)}$ and $F_{ab} = \{xy | xy \text{ edge in } E(G) \text{ such that } \alpha^i(x) = 0, \alpha^i(y) = 1 \text{ and } \alpha^j(x) = \alpha^j(y), \text{ for all } j \neq i\}$. 
Note that the vertices of \( Q_h(111) \) define an isometric embedding \( \alpha \) into \( Q_h \) in a natural way, i.e. if \( u \in V(Q_h(111)) \), then \( \alpha^i(u) = u_i \).

If \( v \) is a vertex of a graph \( G \), then \( N_G(v) \) denotes the set of vertices of \( G \) adjacent to \( v \).

For \( X \subseteq V(G) \) let \( G[X] \) denote the subgraph of \( G \) induced by the set \( X \).

Let \( 0^h \) stand for the vertex with zero at all coordinates and \( 0^{-1}10^{h-i} \) for the vertex with one exactly at the \( i-th \) coordinate for \( i \in [h] \). We write \( xy \) for the concatenation of binary strings \( x \) and \( y \). We will also denote by \( e_i = 0^{-1}10^{h-i} \) the \( i-th \) unit string in \( \{0,1\}^h \).

For binary strings \( u \) and \( v \) of equal length let \( u + v \) denote their sum computed bitwise modulo 2. In particular, \( u + e_i \) is the string obtained from \( u \) by complementing its \( i-th \) bit.

3 Characterization

Let \( ab \) be an edge of a partial cube \( G \) for which \( U_{ab} = W_{ab} \). Then \( G[W_{ab}] \) is called a peripheral subgraph of \( G \).

A \( \Theta \)-class \( E \) of a partial cube \( G \) is called peripheral if at least one of \( G[W_{ab}] \) and \( G[W_{ba}] \) is peripheral for all \( ab \in E \). It is known that every \( \Theta \)-class of a Fibonacci cube is peripheral [Taranenko and Vesel (2007)]. Here we show that the same holds for the 3rd order generalized Fibonacci cubes.

**Lemma 3.1** Every \( \Theta \)-class of \( Q_h(111) \) is peripheral.

**Proof:** Since \( Q_h(111) \) isometrically embeds into \( Q_h \), it admits \( \Theta \)-classes denoted \( 1, \ldots, h \). Note that an edge \( e = uv \) belongs to a \( \Theta \)-class \( i \) if and only if \( u \) and \( v \) differ exactly in the \( i-th \) coordinate. Suppose w.l.o.g. that \( u \in W(i,1) \) and \( v \in W(i,0) \). Let \( x \) be a vertex of \( W(i,1) \) and let \( x' \) be obtained from \( x \) by changing the \( i-th \) position of \( x \) to 0. Since it is straightforward to see that \( x' \) is a vertex of \( W_{uv} \), it follows that \( xx' \in F_{uv} \) and \( W_{uv} = W(i,1) = U_{uv} \). This assertion completes the proof. \( \square \)

Let \( f_h, h \geq 1 \), denote the number of the 3rd order Fibonacci strings of length \( h \), i.e. \( f_h = F_{h+3}^3 = |V(Q_h(111))| \). For convenience, we also set \( f_0 := 1 \).

The following result shows the recursive structure of \( Q_h(111) \) (see also Fig. 2).

**Proposition 3.2** If \( W(i,\chi) \), \( \chi \in \{0,1\} \), is a semicube of \( Q_h(111) \), then the following hold.

(i) \( Q_h(111)[W(i,1)] \) is isomorphic to \( Q_{h-1}(111) \).

(ii) \( |W(i,1)| = f_{h-2} + f_{h-3} \).

(iii) \( |W(i,0) \cap W(2,1)| = f_{h-3} + f_{h-4} \).

(iv) \( Q_h(111)[W(i,1) \cap W(2,1)] \) is isomorphic to \( Q_{h-3}(111) \).

(v) \( Q_h(111)[W(i,1) \cap W(2,0)] \) is isomorphic to \( Q_{h-2}(111) \).

(vi) \( |W(i,0)| = f_{i-1}f_{h-i}, i \in [h] \).

(vii) If \( a = 0^h \) and \( b = 10^{h-1} \), then \( W_{ab} \setminus U_{ab} = W(i,0) \cap W(2,1) \cap W(3,1) \).

**Proof:** It is obvious that \( u \in V(Q_{h-1}(111)) \) if and only if \( 0u \in W(1,0) \). Moreover, \( uv \in E(Q_{h-1}(111)) \) if and only if \( 0u \) and \( 0v \) are adjacent in \( Q_h(111) \). Since \( |V(Q_{h-1}(111))| = f_{h-1} \) and \( |V(Q_h(111))| = f_h = f_{h-1} + f_{h-2} + f_{h-3} \), (i) and (ii) easily follow. Analogously, (iii) follows from (i) and (ii).

For (iv) note that if \( u \) is a vertex of \( W(1,1) \cap W(2,1) \), then \( u_1 = u_2 = 1 \) and \( u_3 = 0 \). Therefore, \( v \in V(Q_{h-3}(111)) \) if and only if \( 110v \in W(1,1) \cap W(2,1) \) and the assertion easily follows. The proof of (v) is analogous.
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For the proof of (vi), let $\mathcal{F}_h$ be the set of all the 3rd order Fibonacci strings of length $h$ and $\mathcal{F}_h^i \subset \mathcal{F}_h$, $i \in [h]$ be the subset which includes all strings of $\mathcal{F}_h$ with 0 at the position $i$. In order to prove the assertion, note that $|\mathcal{F}_h| = f_h$ and $|\mathcal{F}_h^i| = |\mathcal{F}_{h-1} \times \mathcal{F}_{h-1}| = f_{h-1}f_{h-1}$.

Finally, the proof of (vii) note that $U_{ab}$ denotes the set of vertices of $W_{ab}$ that are adjacent to the vertices of $W_{ba}$. We have $W_{ab} = W_{(1,0)}$ and $W_{ba} = W_{(1,1)} \cap (W_{(2,0)} \cup (W_{(2,1)} \cap W_{(3,0)}))$. Since every $\Theta$-class of $Q_h(111)$ is peripheral by Lemma 3.1, we have $W_{ba} = U_{ba}$. It follows that $U_{ab} = W_{(1,0)} \cap ((W_{(2,0)} \cup (W_{(2,1)} \cap W_{(3,0)}))$. Since $W_{(1,0)} = W_{(1,0)} \cap ((W_{(2,0)} \cup (W_{(2,1)} \cap W_{(3,0)}))$, we have $W_{ab} \setminus U_{ab} = W_{(1,0)} \setminus ((W_{(2,0)} \cup (W_{(2,1)} \cap W_{(3,0))})) = W_{(1,0)} \cap W_{(2,1)} \cap W_{(3,1)}$ and the assertion follows.

$Q_h(111)[W_{ab}] = Q_h(111)|W_{ab}| = Q_h(111)$ \quad $W_{ba} = U_{ba} = V(Q_{h-2}(111)) \cup V(Q_{h-3}(111))$

Fig. 2: The structure of $Q_h(111)$

**Lemma 3.3** Let $h \geq 9$ and $x = 10^{h-1} \in V(Q_h(111))$. If $xy$ is an edge of $Q_h(111)[W_{(1,1)}]$ and the set $W_{(1,1)} \cap W_{(2,1)}$ admits exactly $f_{h-3}$ vertices, then $W_{(1,1)} \cap W_{(3,0)} = W_{(1,1)} \cap W_{(2,1)}$.

**Proof:** By Proposition 3.2(ii) we have $|W_{(1,1)}| = f_{h-2} + f_{h-3}$. Moreover, by Lemma 2.3 a vertex $u$ of $W_{(1,1)} \cap W_{(3,0)}$ is adjacent to $v$ of $W_{(1,1)} \cap W_{(2,1)}$ if and only if $u$ and $v$ differ in precisely the $i$-th coordinate, $i \geq 2$. Since $x = 10^{h-1}$, the $i$-th coordinate of $u$ and $v$ is 1 and 0, respectively. From the proof of Lemma 3.1 then it follows that $|W_{xy}| < |W_{yx}|$ and therefore we get $|W_{(1,1)} \cap W_{xy}| < |W_{(1,1)} \cap W_{yx}|$.

If $i = 2$, then $W_{(1,1)} \cap W_{yx} = W_{(1,1)} \cap W_{(2,1)}$. From Proposition 3.2(iv) (see also Fig. 2) where $b$ and $e$ can be replaced with $x$ and $y$, respectively) we can see that $Q_h(111) \cap W_{xy}$ is isomorphic to $Q_h(111)$. Thus, $|W_{(1,1)} \cap W_{xy}| = f_{h-2}$ and $|W_{(1,1)} \cap W_{yx}| = f_{h-3}$.

We will show that $|W_{(1,1)} \cap W_{(i,0)}| \neq f_{h-2}$ for every $i > 2$.

Let $i = 3$. If $u \in W_{(1,1)} \cap W_{(3,0)}$, then either $u_2 = 0$ or $u_2 = 1$. It follows from Proposition 3.2(vi) that $|W_{(1,1)} \cap W_{(3,0)}| = 2f_{h-3} > f_{h-2}$ and the case is settled.
Let \( i = 4 \). If \( u \in W_{(1, 1)} \cap W_{(4, 0)} \), then either \( u_2 = 0 \) or \( u_2 = 1 \) and \( u_3 = 0 \). From Proposition \[5.2\] (vi) it follows that \( |W_{(1, 1)} \cap W_{(4, 0)}| = 3f_{h-4} - 2f_{h-5} + f_{h-6} + f_{h-7} = f_{h-3} + f_{h-4} + f_{h-7} = f_{h-2} - f_{h-5} + f_{h-7} < f_{h-2} \).

Let \( i = h \). If \( u \in W_{(1, 1)} \cap W_{(h, 0)} \), then either \( u_2 = 1 \) and \( u_3 = 0 \) or \( u_2 = 0 \). From Proposition \[5.2\] (vi) it follows that \( |W_{(1, 1)} \cap W_{(h, 0)}| = f_{h-3} + f_{h-4} < f_{h-2} \).

Let \( i = h - 1 \). If \( u \in W_{(1, 1)} \cap W_{(h-1, 0)} \), then either \( u_2 = 1 \) and \( u_3 = 0 \) or \( u_2 = 0 \). Moreover, \( u_{h} = 0 \) or \( u_{h} = 1 \). From Proposition \[5.2\] (vi) it follows that \( |W_{(1, 1)} \cap W_{(h-1, 0)}| = 2(f_{h-4} + f_{h-5}) = f_{h-4} + f_{h-5} + f_{h-3} - f_{h-6} = f_{h-2} - f_{h-6} < f_{h-2} \).

Let \( i = h - 2 \). Similarly as above we get \( |W_{(1, 1)} \cap W_{(h-2, 0)}| = 4(f_{h-5} + f_{h-6}) = 2(f_{h-3} - f_{h-7}) < 2f_{h-3} - f_{h-6} = f_{h-3} + f_{h-3} - f_{h-6} = f_{h-3} + f_{h-4} + f_{h-5} = f_{h-2} \).

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**Tab. 1:** Values of \( |W_{(1, 1)} \cap W_{(i, 0)}| \) for \( i \in \{2, 5, \ldots, h - 3\} \), \( h = \{9, 10, 11\} \).

For \( 5 \leq i \leq h - 3 \) note that for \( u \in W_{(1, 1)} \cap W_{(i, 0)} \) we have either \( u_2 = 0 \) or \( u_2 = 1 \) and \( u_3 = 0 \). From Proposition \[5.2\] (vi) it then follows that \( |W_{(1, 1)} \cap W_{(i, 0)}| = f_{i-3}f_{h-i} + f_{i-4}f_{h-i} \). We will show that \( (f_{i-3} + f_{i-4})f_{h-i} < f_{h-2} \), \( 5 \leq i \leq h - 3 \), by induction on \( h \). For \( h \in \{9, 10, 11\} \) the values of \( |W_{(1, 1)} \cap W_{(i, 0)}| = (f_{i-3} + f_{i-4})f_{h-i} \) are depicted in Table 1 and we can see that \( (f_{i-3} + f_{i-4})f_{h-i} < f_{h-2} \) for every \( i \) of the interest. Suppose that the assertion holds for all dimensions smaller than \( h \), for some \( h > 11 \). Let us consider the dimension \( h \) for \( 5 \leq i \leq h - 3 \). We have \( (f_{i-3} + f_{i-4})f_{h-i} = (f_{i-3} + f_{i-4})f_{h-i-1} + (f_{i-3} + f_{i-4})f_{h-2-i} + (f_{i-3} + f_{i-4})f_{h-3-i} \). Note that by the induction hypothesis:

\[ (f_{i-3} + f_{i-4})f_{h-i-1} < f_{h-3}, \quad 5 \leq i \leq h - 1, \]

\[ (f_{i-3} + f_{i-4})f_{h-2-i} < f_{h-4}, \quad 5 \leq i \leq h - 2, \]

\[ (f_{i-3} + f_{i-4})f_{h-3-i} < f_{h-5}, \quad 5 \leq i \leq h - 3. \]

It follows that \( (f_{i-3} + f_{i-4})f_{h-i} < f_{h-3} + f_{h-4} + f_{h-5} = f_{h-2}, \quad 5 \leq i \leq h - 3. \)

Since we showed above that \( (f_{i-3} + f_{i-4})f_{h-i} < f_{h-2} \) for \( i \in \{h-2, h-1, h\} \), the proof is complete.

\( \square \)

**Lemma 3.4** Let \( xy \) be an edge of \( Q_h(111)[W_{(1, 1)}] \) such that \( x = 10^{h-1} \). If \( H := Q_h(111)[W_{(1, 1)} \cap W_{(g, 1)}] \), then the following hold.

(i) If \( h \in \{7, 8\} \), then \( |V(H)| = f_{h-3} \) if and only if \( V(H) = W_{(1, 1)} \cap W_{(2, 1)} \) or \( V(H) = W_{(1, 1)} \cap W_{(h-2, 1)} \). Moreover, if \( V(H) = W_{(1, 1)} \cap W_{(2, 1)} \), then \( H \) is isomorphic to \( Q_{h-3}(111) \).

(ii) If \( h \in \{5, 6\} \), then \( |V(H)| = f_{h-3} \) if and only if \( V(H) = W_{(1, 1)} \cap W_{(2, 1)} \).

(iii) If \( h \in \{3, 4\} \), then \( |V(H)| = f_{h-3} \) if and only if \( V(H) = W_{(1, 1)} \cap W_{(2, 1)} \) or \( V(H) = W_{(1, 1)} \cap W_{(3, 1)} \). Moreover, if \( V(H) = W_{(1, 1)} \cap W_{(3, 1)} \), then \( H \) is isomorphic to \( Q_{h-3}(111) \).

**Proof:** Note that in the graph \( Q_h(111) \) we have \( |W_{(1, 1)}| = |W_{(1, 1)} \cap W_{(i, 0)}| + |W_{(1, 1)} \cap W_{(i, 1)}| = f_{h-2} + f_{h-3}, \quad i \in \{2, \ldots, h\} \). By Proposition \[5.2\] (ii). We can obtain analogously as in the proof of Lemma \[3.3\] the number of vertices of \( W_{(1, 1)} \cap W_{(i, 0)} \) and \( W_{(1, 1)} \cap W_{(i, 1)} \) as follows. If \( i = 2 \), then...
| \(W_{(1,1)} \cap W_{(i,0)}\) = \(f_h - 2\) and \(|W_{(1,1)} \cap W_{(i,1)}| = f_h - 3\); if \(i = 3\), then \(|W_{(1,1)} \cap W_{(i,0)}| = 2f_{h-3}\); for \(i \in \{4, \ldots, h\}\) we get \(|W_{(1,1)} \cap W_{(i,0)}| = f_h - 3f_{h-1} + f_{h-3}f_{h-1}\). The values of \(|W_{(1,1)} \cap W_{(i,0)}|\) for every \(i \in \{2, \ldots, h\}\) and for all \(h = \{5, 6, 7, 8\}\) are depicted in Table 2.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
h \setminus i & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
5 & 7 & 8 & 6 & 6 & & & \ \\
6 & 13 & 14 & 12 & 11 & & & \ \\
7 & 24 & 26 & 21 & 24 & 20 & & \ \\
8 & 44 & 48 & 39 & 42 & 44 & 40 & 37 \ \\
\hline
\end{array}
\]

Tab. 2: Values of \(|W_{(1,1)} \cap W_{(i,0)}|\) for every \(i \in \{2, \ldots, h\}\) and for all \(h = \{5, 6, 7, 8\}\).

(i) Since \(f_5 = 24\) and \(f_6 = 44\), the first part of the assertion follows from Table 2. In order to see that \(Q_7(111)(|W_{(1,1)} \cap W_{(5,1)}|\) is not isomorphic to \(Q_4(111)\), observe first that \(V(Q_7(111)(|W_{(1,1)} \cap W_{(5,1)}|)| = \{1000100, 1000101, 1000110, 1001000, 1001001, 1001010, 1001011, 1001101, 1001110, 1010100, 1010101, 1100100\}.

Note that by Proposition 3.2, the graph isomorphic to \(Q_4(111)\) has to admit an edge \(ab\) such that \(|W_{ab}| = f_4 - 1 = f_3 = 7\) and \(|W_{ba}| = f_2 + f_1 = 6\).

We can see that \(|W_{(1,1)} \cap W_{(5,1)}| = 5\), \(|W_{(1,1)} \cap W_{(5,1)} \cap W_{(3,1)}| = 3\), \(|W_{(1,1)} \cap W_{(5,1)} \cap W_{(4,1)}| = 4\), \(|W_{(1,1)} \cap W_{(5,1)} \cap W_{(6,1)}| = 3\), \(|W_{(1,1)} \cap W_{(5,1)} \cap W_{(7,1)}| = 5\). It follows that \(Q_7(111)(|W_{(1,1)} \cap W_{(5,1)}|\) cannot be isomorphic to \(Q_4(111)\) and the case is settled.

By Proposition 3.2, the graph isomorphic to \(Q_5(111)\) has to admit an edge \(ab\) such that \(|W_{ab}| = f_5 - 1 = f_4 = 13\) and \(|W_{ba}| = f_3 + f_2 = 11\).

We can see that \(|W_{(1,1)} \cap W_{(6,1)} \cap W_{(2,1)}| = 8\), \(|W_{(1,1)} \cap W_{(6,1)} \cap W_{(3,1)}| = 8\), \(|W_{(1,1)} \cap W_{(6,1)} \cap W_{(4,1)}| = 9\), \(|W_{(1,1)} \cap W_{(6,1)} \cap W_{(5,1)}| = 6\), \(|W_{(1,1)} \cap W_{(6,1)} \cap W_{(7,1)}| = 6\), \(|W_{(1,1)} \cap W_{(6,1)} \cap W_{(8,1)}| = 9\). It follows that \(Q_8(111)(|W_{(1,1)} \cap W_{(6,1)}|\) cannot be isomorphic to \(Q_5(111)\) and the case is settled.

(ii) Since we can see from Table 2 that \(|W_{(1,1)} \cap W_{(i,0)}| = f_{h-2}\) only if \(i = 2\), the assertion follows.

(iii) The assertion follows from Fig. 4. \(\square\)

**Remark 3.5** The results of Proposition 3.2, Lemmas 3.3 and 3.4, can be analogously stated for \(b = 0^{h-1}, W_{(h-1),}\) and \(x = 0^{h-1}\). In particular, an analog to Lemma 3.3 is as follows:

**Let** \(h \geq 9\) **and** \(x = 0^{h-1} \in V(Q_{h}(111))\). **If** \(xy\) **is an edge of** \(Q_{h}(111)(|W_{(h,1)}|)\) **and the set** \(W_{(h,1)} \cap W_{y^e}\) **admits exactly** \(f_{h-3}\) **vertices, then** \(W_{(h,1)} \cap W_{y^e} = W_{(h,1)} \cap W_{(h-1,1)}\).

**Lemma 3.6** Let \(h \geq 3\) **and let** \(v\) **be a vertex of** \(Q_{h}(111)\). **Then** \(\deg(v) < h\) **if**

(i) \(v_i = v_{i+1} = 1, i \leq h - 1\), or

(ii) \(v_i = v_{i+2} = 1, i \leq h - 2\).

**Proof:** (i) Since \(v\) cannot have three consecutive ones and \(h \geq 3\), we have \(v_{i-1} = 0\) or \(v_{i+2} = 0\). Suppose w.l.o.g. that \(v_{i-1} = 0\). Let \(u \in N(v)\). Note that \(v\) and \(u\) differ in precisely one coordinate. Since
\[ v_i = v_{i+1} = 1 \text{ and } v_{i-1} = 0, \text{ and } \nu \text{ cannot differ in the } (i-1)\text{-st coordinate and the case is settled. The proof for (ii) is analogous.} \]

**Proposition 3.7** (i) If \( h \geq 3 \), then \( 0^h \) is the only vertex with \( h \) neighbors of degree \( h \) in \( Q_h(111) \).

(ii) Let \( h \geq 4 \) and \( v = 0^{i-1}10^{h-i-1}, i \in [h] \). Then \( v \) has \( h - 2 \) neighbors of degree \( h \) in \( Q_h(111) \) if and only if \( i = 1 \) or \( i = h \).

**Proof:**

(i) It is clear that \( 0^h \) has \( h \) neighbors of degree \( h \) in \( Q_h(111) \). Assume to the contrary that \( Q_h(111) \) admits a vertex \( v \) with \( h \) neighbors of degree \( h \) such that \( v_i = 1 \). Note that \( v \) cannot have \( v_i = v_{i+1} = 1, i \leq h - 1 \), or \( v_i = v_{i+2} = 1, i \leq h - 2 \), by Lemma 3.6. It follows that there exist a vertex \( u \) adjacent to \( v \) such that either \( u_{i+1} = 1 \) or \( u_{i-1} = 1 \). But then we have either \( u_i = u_{i+1} = 1 \) or \( u_{i-1} = u_i = 1 \). From Lemma 3.6 it follows that \( u \) admits less then \( h \) neighbors and we obtain a contradiction.

(ii) Note that \( 10^{h-1} \) and \( 0^{h-1} \) have \( h - 2 \) neighbors of degree \( h \) in \( Q_h(111) \). Assume to the contrary that \( Q_h(111) \) admits a vertex \( v = 0^i10^{h-i-1} \) with exactly \( h - 2 \) neighbors of degree \( h \) such that \( 2 \leq i \leq h - 1 \). We can see from Lemma 3.6 that \( v + e_{i+1} \) and \( v + e_{i-1} \) are neighbors of \( v \) with a degree less then \( h \). Moreover, since \( h \geq 4 \), at least one of \( v + e_{i+2} \) and \( v + e_{i-2} \) exists in \( Q_h(111) \). Since both have less then \( h \) neighbors in \( Q_h(111) \) by Lemma 3.6, we again obtain a contradiction.

If \( G \) is an isometric subgraph of \( Q_h \), let \( G^{i,j}, i,j \in [h] \), denote a graph obtained from \( G \) by exchanging the \( i \)-th and the \( j \)-th coordinate in every vertex of \( V(G) \). Note that \( G^{i,j} \) is an isometric subgraph of \( Q_h \) isomorphic to \( G \). From Fig. 1 we can see

**Proposition 3.8** The vertices of \( Q_3(111)[W_{(1,1)}]^{2,3} \) and \( Q_4(111)[W_{(1,1)}]^{2,3} \) are the 3rd order Fibonacci strings.

The next theorem gives a new characterization of \( Q_h(111) \).

**Theorem 3.9** Let \( ab \) be an edge of a connected, bipartite graph \( G \) such that \( a \) has \( h \) neighbors of degree \( h \) and \( b \) has \( h - 2 \) neighbors of degree \( h \).

If \( h \geq 10 \) or \( h \in \{4, 5, 6, 7\} \), then \( G \) is isomorphic to \( Q_h(111) \) if and only if the following conditions hold:

(i) \( G[W_{ab}] \) is isomorphic to \( Q_{h-1}(111) \).

(ii) \( F_{ab} \) is a matching defining an isomorphism between \( G[U_{ab}] \) and \( G[W_{ba}] \).

(iii) \( W_{ab} \setminus U_{ab} = W_{ab} \cap W_{ca} \cap W_{dc} \) for \( c \in U_{ab} \) which has \( h - 3 \) neighbors of degree \( h - 1 \) in \( G[W_{ab}] \) and is adjacent to a vertex \( d \in W_{ab} \setminus U_{ab} \).

(iv) \( |W_{ab} \setminus U_{ab}| = f_{h-4} \).

**Proof:** Let \( G \) be isomorphic to \( Q_h(111) \). From Proposition 3.7 it follows that \( a = 0^h \) and \( b = 10^{h-1} \) or \( b = 0^{h-1} \). Suppose first that \( b = 10^{h-1} \).

(i) This is Proposition 3.2(i).

(ii) Since \( a = 0^h \) and \( b = 10^{h-1} \), we have \( W_{ab} = W_{(1,0)} \) and \( W_{ba} = W_{(1,1)} \). Thus, an edge of \( F_{ab} \) has one end-vertex with zero at the first position and one end-vertex with one at the first position. Since all \( \Theta \)-classes of \( Q_h(111) \) are peripheral by Lemma 3.1 it follows that \( W_{ba} = U_{ba} \). Let \( u, v \in W_{ba} \)
and \( u', v' \in W_{ab} \) such that \( uu', vv' \in F_{ab} \). Since \( u \) and \( u' \) (resp. \( v \) and \( v' \)) differ precisely in the first coordinate, \( uu \in E(G[W_{ab}]) \) if and only if \( uu' \in E(G[U_{ab}]) \). It follows that \( G[U_{ab}] \) and \( G[W_{ba}] \) are isomorphic.

(iii) From (i) and Proposition 3.7 it follows that \( c = 010^h-2 \) or \( c = 00^h-21 \). Since \( b = 10^h-1 \), Proposition 3.2(vii) yields \( W_{ab} \setminus U_{ab} = W_{(1,0)} \cap W_{(2,1)} \cap W_{(3,1)} \), hence \( c = 010^h-2 \) and \( d = 0110^h-3 \).

(iv) Every \( u \in W_{ab} \setminus U_{ab} \) has the first four coordinates fixed such that \( u_4 = 0 \). Hence, one can construct all the 3rd order Fibonacci strings of length \( h - 4 \) at the coordinates 5, 6, \ldots, \( h \) and we have \( |W_{ab} \setminus U_{ab}| = f_{h-4} \).

If the proof for \( b = 0^h-1 \) is analogous (see also Remark 3.5), the first part of the proof is complete.

For the converse note that \( G[W_{ab}] \) is isomorphic to \( Q_{h-1}(111) \), thus we will first construct an isometric embedding of \( G[W_{ab}] \) into \( Q_{h-1}(111) \), i.e., we will assign the 3rd order Fibonacci strings of length \( h - 1 \) to the vertices of \( G[W_{ab}] \). Denote this embedding by \( \beta \). Since \( |N_G(a)| = h \) and \( G \) is bipartite, we have \( |N_G[W_{ab}](a)| \) is an embedding into \( Q_{h-1}(111) \), the degree of a vertex of \( G[W_{ab}] \) cannot exceed \( h - 1 \). Hence, \( \alpha \) has \( h - 1 \) neighbors of degree \( h - 1 \) in \( G[W_{ab}] \). From Proposition 3.7, it then follows that we have to set \( \beta(a) := (0, 0, \ldots, 0) \). Analogously, since \( c \) has \( h - 3 \) neighbors of degree \( h - 1 \) in \( G[W_{ab}] \), we may set \( \beta(c) := (1, 0, 0, \ldots, 0) \) (we could also set \( \beta(c) := (0, 0, 0, \ldots, 0, 1) \) by Proposition 3.7). We may assume that \( \beta(c) := (1, 0, 0, \ldots, 0) \).

It follows that for every \( u \in W_{ab} \cap W_{ac} \) we have \( \beta^1(u) = 0 \) and for every \( v \in W_{ab} \cap W_{ca} \) we have \( \beta^3(v) = 1 \).

Suppose first that \( h \geq 10 \). Note that \( G[W_{ab}] \) is isomorphic to \( Q_{h-1}(111) \), \( d \) belongs to \( W_{ab} \cap W_{ca} \), \( W_{ab} \cap W_{ca} \cap W_{de} = W_{ab} \setminus U_{ab} \) and \( |W_{ab} \setminus U_{ab}| = f_{h-4} \). Hence, by Lemma 3.3, we have to define \( \beta \) such that

\[
\text{for every vertex } u \in W_{ab} \setminus U_{ab}, \quad \beta^3(u) = 1 \quad \text{and} \quad \beta^1(u) = 0.
\]

If \( h \in \{4, 5, 6, 7\} \) note that since \( \beta \) is an embedding into \( Q_{h-1}(111) \), the value of \( h \) is subtracted by 1 when it is applied in Lemma 3.4 and Proposition 3.8. If \( h \in \{6, 7\} \), then by Lemma 3.4(ii), we have to define \( \beta \) such that for every vertex \( u \in W_{ab} \setminus U_{ab} \) we have \( \beta^3(u) = 1 \) and \( \beta^1(u) = 0 \). If \( h \in \{4, 5\} \), then by Lemma 3.4(iii), for every vertex \( u \in W_{ab} \setminus U_{ab} \) we have either \( \beta^3(u) = 1 \) and \( \beta^1(u) = 0 \) or \( \beta^3(u) = 0 \) and \( \beta^1(u) = 1 \). From Proposition 3.5 it follows that above both cases are equivalent, therefore we may choose the former and conclude that for \( h \geq 10 \) or \( h \in \{4, 5, 6, 7\} \) for every vertex \( u \in W_{ab} \cap W_{ca} \cap W_{de} \) we have \( \beta^3(u) = 1 \) and \( \beta^1(u) = 0 \), while for every vertex \( v \in W_{ab} \cap W_{ca} \cap W_{cd} \) we have \( \beta^3(v) = 1 \) and \( \beta^1(v) = 0 \). From Proposition 3.3 and Proposition 3.2(iv) imply that the vertices of \( W_{ab} \setminus U_{ab} \) induce \( Q_{h-4}(111) \).

We may then construct \( \beta \) for \( u \in W_{ab} \cap W_{ca} \cap W_{de} \) such that \( (\beta^3(u), \beta^5(u), \ldots, \beta^{2h-1}(u)) \) represents the embedding of \( Q_{h}(111)[W_{ab} \setminus U_{ab}] \) into \( Q_{h-4}(111) \) and the vertices of \( W_{ab} \cap W_{ca} \cap W_{cd} \) accordingly. Finally, we construct \( \beta \) for every vertex of \( W_{ab} \cap W_{ac} \) with respect to the embedding of vertices of \( W_{ab} \cap W_{ca} \) and the construction of \( \beta \) is complete.

We can now apply \( \beta \) in order to construct an embedding \( \alpha \) of vertices of \( G \) into \( Q_h(111) \) as follows.

For every vertex \( x \in W_{ab} \) we set \( \alpha^1(x) := 0 \) and \( \alpha^i(x) := \beta^{i-1}(x), 2 \leq i \leq h \). Note that for every vertex \( y \in W_{ba} \) there exist a vertex \( x \in W_{ab} \) such that \( xy \in F_{ab} \). Thus, we set \( \alpha^1(y) := 1 \) and \( \alpha^i(y) := \beta^{i-1}(x), 2 \leq i \leq h \).

Obviously, for every vertex \( v \) of \( G \) we constructed the embedding \( \alpha \) such that \( \alpha^1(v) \alpha^2(v) \ldots \alpha^h(v) \) is a 3rd order Fibonacci string. In order to conclude the proof, we have to show that for every \( v \) we have \( uv \in E(G) \) if and only if \( \alpha(u) \alpha(v) \in E(Q_h(111)) \). Since \( G[W_{ab}] \) is isomorphic to \( Q_{h-1}(111) \), the claim is obvious for every \( u, v \in W_{ab} \). For \( u, v \in W_{ba} \) note that since the matching \( F_{ab} \) defines an isomorphism between \( G[W_{ba}] \) and \( G[U_{ab}] \), for every \( u, v \in W_{ba} \) we have \( u', v' \in U_{ab} \) such that \( uu', vv' \in F_{ab} \) and
proof of Theorem 3.9.

Procedure GENERALIZED3 FIBONACCI($G$, $h$):

1. If $h \leq 3$ and $G$ is isomorphic to one of \{ $Q_1$, $Q_2$, $Q_3$ \} then ACCEPT.
2. Find ab $\in E(G)$ such that a has $h$ neighbors of degree $h$ and $b$ has $h-2$ neighbors of degree $h$.
3. If no such edge exists then REJECT.
4. Find the sets $W_{ab}$, $W_{ba}$, $U_{ab}$, $U_{ba}$ and $F_{ab}$.
5. Construct the graph $G[W_{ab}]$.
6. Find ac $\in E(G[W_{ab}])$ such that c has $h-3$ neighbors of degree $h-1$ in $G[W_{ab}]$ and c is adjacent to a vertex $d \in W_{ab} \setminus U_{ab}$.
7. If no such edge exists then REJECT else find sets $W_{ac}$, $W_{ca}$, $W_{dc}$.
8. Verify that
   8.1. GENERALIZED3 FIBONACCI($G[W_{ab}]$, $h-1$) returns ACCEPT.
   8.2. $F_{ab}$ is a matching which defines an isomorphism between $G[U_{ab}]$ and $G[W_{ba}]$.
   8.3. $W_{ab} \setminus U_{ab} = W_{ab} \cap W_{ca} \cap W_{dc}$.
   8.4. $|W_{ab} \setminus U_{ab}| = f_{h-4}$.
   8.5. $h \not\in \{8, 9\}$ or $G[W_{ab} \setminus U_{ab}]$ is isomorphic to $Q_{h-4}(111)$.
9. If all of the foregoing conditions are fulfilled then ACCEPT else REJECT.

Since Theorem 3.9 does not include the graphs $Q_h(111)$ for $h \in \{8, 9\}$, we need the following

Proposition 3.10 Let ab be an edge of a connected, bipartite graph $G$ such that $a$ has $h$ neighbors of degree $h$, $b$ has $h-2$ neighbors of degree $h$.

If $h \in \{8, 9\}$, then $G$ is isomorphic to $Q_h(111)$ if and only if the following conditions hold:

(i) $G[W_{ab}]$ is isomorphic to $Q_{h-1}(111)$.

(ii) $F_{ab}$ is a matching defining an isomorphism between $G[U_{ab}]$ and $G[W_{ba}]$.

(iii) $W_{ab} \setminus U_{ab} = W_{ab} \cap W_{ca} \cap W_{dc}$ for $c \in U_{ab}$ which has $h-3$ neighbors of degree $h-1$ in $G[W_{ab}]$ and is adjacent to a vertex $d \in W_{ab} \setminus U_{ab}$.

(iv) $|W_{ab} \setminus U_{ab}| = f_{h-4}$.

(v) $G[W_{ab} \setminus U_{ab}]$ is isomorphic to $Q_{h-4}(111)$.

Proof: For the if part of the proof just note that by Lemma 3.4(i), for $h \in \{8, 9\}$ the conditions (i)-(iv) do not guarantee that $G[W_{ab} \setminus U_{ab}]$ is isomorphic to $Q_{h-4}(111)$. The rest of the proof is analogous to the proof of Theorem 3.9.
Before GENERALIZED\textsuperscript{3} FIBONACCI is started, some preprocessing has to be done. A given graph $G$ is examined only if $|V(G)| = F_{h+3}^3$, for some $h \geq 1$, otherwise the graph is rejected. Moreover, we have to establish whether $G$ is bipartite. Note that this can be done in $O(m)$ time.

The given bipartite graph $G$ is then declared the 3rd order generalized Fibonacci cube if and only if GENERALIZED\textsuperscript{3} FIBONACCI($G,h$) terminates without ever encountering a REJECT statement. In other words, GENERALIZED\textsuperscript{3} FIBONACCI for a graph $G$ is examined only if $|^G| = F_{h+3}^3$, and the conditions of Theorem 3.9 and Proposition 3.10 are fulfilled for $G$ or $h \leq 3$ and $G$ is isomorphic to one of $\{Q_1, Q_2, Q_3\}$. This gives us the following

**Theorem 4.1** GENERALIZED\textsuperscript{3} FIBONACCI correctly recognizes $Q_h(111)$.

In order to find the time complexity of the algorithm, we first show two lemmas.

**Lemma 4.2** If $h \geq 3$, then

$$
\sum_{i=1}^{h} i f_i = \frac{f_1 + f_2 + (h-1)(f_{h+2} + f_h) - f_{h-1} - f_{h-2}}{2}.
$$

**Proof:** We first compute $2 \sum_{i=1}^{h} i f_i = \sum_{i=1}^{h} 2i f_i$ such that we write every $2i f_i$ as $i f_i + i(f_{i+3} - f_{i+2} - f_{i+1})$.

$$
2 \sum_{i=1}^{h} i f_i = h f_h + h(f_{h+3} - f_{h+2} - f_{h+1}) + (h-1)f_{h-1} + (h-1)(f_{h+2} - f_{h+1} - f_h) +
$$

$$
= h f_{h+3} - f_{h+2} - (h+1)f_{h+1} + \ldots + if_i - (i-1)f_i - (i-2)f_i + (i-3)f_i + \ldots + f_2 + f_1
$$

We obtain above telescoping series in which all terms $i - (i-1)f_i - (i-2)f_i + (i-3)f_i, i = 3, \ldots, h$, are cancelled. Thus we get

$$
2 \sum_{i=1}^{h} i f_i = f_1 + f_2 + h f_{h+3} - f_{h+2} - (h+1)f_{h+1}
$$

$$
= f_1 + f_2 + h f_{h+2} + h f_{h+1} + h f_h - f_{h+2} - (h+1)f_{h+1}
$$

$$
= f_1 + f_2 + h(f_{h+2} + f_h) - f_{h+2} - f_{h+1}
$$

$$
= f_1 + f_2 + (h-1)(f_{h+2} + f_h) - f_{h-1} - f_{h-2}.
$$

The assertion now clearly follows. \hfill \Box

**Theorem 4.3** GENERALIZED\textsuperscript{3} FIBONACCI runs in linear time.
Proof: Let \( m \) denote the number of edges of \( G \).

Concerning the time complexity, we will show that the time complexity of every step of the algorithm with the exception of Step 8.1 is bounded by \( O(m) \). Since for \( h \leq 3 \) the graphs \( Q_h = Q_h(111) \) are very simple, this is obviously true for Step 1. For Steps 2 and 3 it is convenient to arrange the vector \( \deg \), such that \( \deg_v \) equals the number of vertices adjacent to \( v \) for every \( v \in V(G) \). We then first determine vertex \( a \) with \( h \) neighbors of degree \( h \), by inspecting the adjacency list for every vertex of a graph. The total number of all examined entries in the adjacency list is clearly bounded by \( O(m) \). If \( a \) is found, then we check its neighbors in order to find the vertex \( b \). Analogously, this can be done again in linear time and the proof for Steps 2 and 3 is settled. Concerning Steps 4, 5, and 7 it has been shown in [Jha and Slutzki (1992)] that they can be performed in time linear in the number of edges of the input graph. Since the proof for Step 6 is analogous to the proof for Steps 2 and 3, we may proceed with Step 8. It has also been shown in [Jha and Slutzki (1992)] that Step 8.2 can be performed in time linear in the number of edges of the input graph. If we mark every vertex of \( U_{ab} \cup U_{ba} \), \( W_{ab} \), \( W_{ca} \), and \( W_{dc} \), then Steps 8.3 and 8.4 can be implemented to run within the same time bound. Thus, neglecting the recursive call, the total time needed to check \( G \) is bounded by \( O(m) \), i.e. the cost per edge processed by a single call of the algorithm is \( O(1) \).

By Lemma 2.2 we have \( m = \Theta(n \log n) \). If follows that we can find a constant \( C \) such the the total time needed to check \( G \), neglecting the recursive call, is bounded by \( C h f_h \).

Let \( m_h \) denote the total number of edges checked by the algorithm.

From the above discussion we have

\[
m_h \leq \sum_{i=1}^{h} C i f_i = C \sum_{i=1}^{h} i f_i,
\]

while Lemma 4.2 yields

\[
m_h \leq C \frac{f_1 + f_2 + (h - 1)(f_{h+2} + f_h) - f_{h-1} - f_{h-2}}{2}
\]

\[
= C \frac{f_1 + f_2 + (h - 1)(3f_h + 2f_{h-1} + f_{h-2}) - f_{h-1} - f_{h-2}}{2}
\]

\[
= C \frac{f_1 + f_2 + 3h f_h - 3f_h + 2h f_{h-1} - 3f_{h-1} + h f_{h-2} - 2f_{h-2}}{2}
\]

\[
= C \frac{f_1 + f_2 + (3h - 3)f_h + (2h - 3)f_{h-1} + (h - 2)f_{h-2}}{2}
\]

\[
\leq C \frac{5h - 6}{2} f_h \leq \frac{5}{2} C h f_h.
\]

We showed that the total number of edges processed by \( \text{GENERALIZED3 FIBONACCI}(G, h) \) cannot exceed \( \frac{5}{2} |E(G)| \). Moreover, since we showed that the cost per edge processed by a single call of the the algorithm is \( O(1) \), the proof is complete. \( \square \)
References


