Joso Vukman

ON (m, n) -JORDAN DERIVATIONS AND COMMUTATIVITY OF PRIME RINGS

Abstract. The purpose of this paper is to prove the following result. Let $m \geq 1, n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with $\text{char}(R) \neq 2mn(m+n)$ and $\text{char}(R) \nmid m-n$. Suppose there exists a nonzero additive mapping $D : R \rightarrow R$ satisfying the relation $(m+n)D(x^2) = 2mD(x)x + 2nxD(x)$ for all $x \in R$ ((m, n) -Jordan derivation). If either $\text{char}(R) = 0$ or $\text{char}(R) > 3$ then D is a derivation and R is commutative.

This research is related to our earlier work [3] and [12]. Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be n -torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. For $x, y \in R$ we write $[y, x]_1 = [y, x] = yx - xy$, and for $n \geq 1$, $[y, x]_n = [[y, x]_{n-1}, x]$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \rightarrow R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$, and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Obviously, any derivation is a Jordan derivation. The converse is in general not true. Herstein [9] has proved that any Jordan derivation on a prime ring with $\text{char}(R) \neq 2$ is a derivation. A brief proof of Herstein's result can be found in [1]. Cusack [7] has proved Herstein's theorem for 2-torsion free semiprime rings (see [2] for an alternative proof). An additive mapping $D : R \rightarrow R$ is called a left derivation if $D(xy) = yD(x) + xD(y)$ holds for all pairs $x, y \in R$ and is called a left Jordan derivation (or Jordan left derivation) in case $D(x^2) = 2xD(x)$ is fulfilled for all $x \in R$. The concepts of left derivation and left Jordan derivation were introduced by Brešar and Vukman in [3]. One can easily prove (see [3]) that the existence of a nonzero

2000 *Mathematics Subject Classification*: 16N60.

Key words and phrases: Prime ring, semiprime ring, Jordan derivation, left derivation, left Jordan derivation, commuting mapping, centralizing mapping.

This research has been supported by the Research Council of Slovenia.

left derivation on a prime ring forces the ring to be commutative. Moreover, we have the following result.

THEOREM 1. *Let R be a prime ring, and let $D : R \rightarrow R$ be a nonzero left Jordan derivation. If $\text{char}(R) \neq 2$ then D is a derivation and R is commutative.*

The result above has been first proved by Brešar and Vukman [3] under the additional assumption that $\text{char}(R) \neq 3$. Later on Deng [8] has proved that the assumption $\text{char}(R) \neq 3$ is superfluous. Theorem 1 is related to the theory of commuting and centralizing mappings. A mapping F , which maps a ring R into itself, is called centralizing on R in case $[F(x), x] \in Z(R)$ holds for all $x \in R$. In a special case when $[F(x), x] = 0$ is fulfilled for all $x \in R$, F is called commuting on R . A classical result of Posner (Posner's second theorem) [10] states that the existence of a nonzero centralizing derivation $D : R \rightarrow R$, where R is a prime ring, forces the ring to be commutative.

We proceed with the following definition.

DEFINITION 1. Let $m \geq 0, n \geq 0$ with $m + n \neq 0$ be some fixed integers. An additive mapping $D : R \rightarrow R$, where R is an arbitrary ring, is called a (m, n) -Jordan derivation in case

$$(1) \quad (m + n)D(x^2) = 2mD(x)x + 2nxD(x)$$

holds for all $x \in R$.

Let us point out that the relation (1) appears naturally in the proof of Theorem 1 in [11]. Obviously, $(1, 1)$ -Jordan derivation on a 2-torsion free ring is a Jordan derivation and $(1, 0)$ -Jordan derivation is a left Jordan derivation.

We proceed with the following proposition.

PROPOSITION 1. *Let $m \geq 0, n \geq 0$ with $m + n \neq 0$ be some fixed integers, let R be a 2-torsion free ring, and let $D : R \rightarrow R$ be an (m, n) -Jordan derivation. In this case the relation*

$$(2) \quad \begin{aligned} (m + n)^2 D(xyx) = & m(n - m)D(x)xy + m(m - n)D(y)x^2 \\ & + n(n - m)x^2 D(y) + n(m - n)yx D(x) \\ & + m(3m + n)D(x)yx + 4mnx D(y)x \\ & + n(3n + m)xy D(x), \quad x, y \in R \end{aligned}$$

holds for all pairs $x, y \in R$.

Proof. The linearization of the relation (1) gives

$$(3) \quad \begin{aligned} (m + n)D(xy + yx) \\ = 2mD(x)y + 2mD(y)x + 2nx D(y) + 2ny D(x), \quad x, y \in R. \end{aligned}$$

Putting in the relation (2) $(m+n)(xy+yx)$ for y , we obtain

$$\begin{aligned} & (m+n)^2 D(x^2y+yx^2) + 2(m+n)^2 D(xyx) \\ &= 2m(m+n)D(x)(xy+yx) + 2m(m+n)D(xy+yx)x \\ & \quad + 2n(m+n)x D(xy+yx) + 2n(m+n)(xy+yx)D(x), \quad x, y \in R. \end{aligned}$$

Applying (2) and (1), we obtain

$$\begin{aligned} & 4m^2 D(x)xy + 4mnxD(x)y + 2m(m+n)D(y)x^2 + 2n(m+n)x^2 D(y) \\ & \quad + 4mnyD(x)x + 4n^2 yxD(x) + 2(m+n)^2 D(xyx) \\ &= 2m(m+n)D(x)(xy+yx) + 4m^2 D(x)yx + 4m^2 D(y)x^2 + 4mnxD(y)x \\ & \quad + 4mnyD(x)x + 4mnxD(x)y + 4mnxD(y)x + 4n^2 x^2 D(y) \\ & \quad + 4n^2 xyD(x) + 2n(m+n)(xy+yx)D(x), \quad x, y \in R. \end{aligned}$$

After collecting terms we obtain

$$\begin{aligned} (m+n)^2 D(xyx) &= m(n-m)D(x)xy + m(m-n)D(y)x^2 \\ & \quad + n(n-m)x^2 D(y) + n(m-n)yx D(x) \\ & \quad + m(3m+n)D(x)yx + 4mnxD(y)x \\ & \quad + n(3n+m)xyD(x), \quad x, y \in R \end{aligned}$$

which completes the proof.

It is our aim in this paper to prove the following result.

THEOREM 2. *Let $m \geq 1, n \geq 1$ be some fixed integers with $m \neq n$ and let R be a prime ring with $\text{char}(R) \neq 2mn(m+n) \mid m-n$. Suppose $D: R \rightarrow R$ is a nonzero (m, n) -Jordan derivation. If $\text{char}(R) = 0$ or $\text{char}(R) > 3$, then D is a derivation and R is commutative.*

The proof of the above theorem depends heavily on the following result proved by Brešar [5] (see also [6]).

THEOREM 3. *Let R be a prime ring and $F: R \rightarrow R$ an additive mapping. Suppose that $[f(x), x]_n = 0$, for all $x \in R$ and some fixed integer $n > 1$. If either $\text{char}(R) = 0$ or $\text{char}(R) > n$, then $[F(x), x] = 0$, for all $x \in R$.*

Proof of Theorem 2. Putting in the relation (2) $y = x$ we obtain

$$\begin{aligned} (4) \quad & (m+n)^2 D(x^3) = m(3m+n)D(x)x^2 \\ & \quad + 4mnxD(x)x + n(3n+m)x^2 D(x), \quad x \in R. \end{aligned}$$

Now, putting $(m+n)^2x^3$ for y in the relation (3) and applying (4) we obtain

$$\begin{aligned}
 (m+n)^3D(x^4) &= m(m+n)^2D(x)x^3 + m(m+n)^2D(x^3)x \\
 &\quad + n(m+n)^2xD(x^3) + n(m+n)^2x^3D(x) \\
 &= m(m+n)^2D(x)x^3 + m((3m^2 + mn)D(x)x^2 \\
 &\quad + 4mnxD(x)x \\
 &\quad + (3n^2 + mn)x^2D(x))xnx((3m^2 + mn)D(x)x^2 \\
 &\quad + 4mnxD(x)x + (3n^2 + mn)x^2D(x)) + n(m+n)^2x^3D(x) \\
 &= (m(m+n)^2 + m(3m^2 + mn))D(x)x^3 + (n(3m^2 + mn) \\
 &\quad + 4m^2n)xD(x)x^2 + (m(3n^2 + mn) + 4mn^2)x^2D(x)x \\
 &\quad + (n(m+n)^2 + n(3n^2 + mn))x^3D(x), \quad x \in R.
 \end{aligned}$$

We have therefore

$$\begin{aligned}
 (5) \quad (m+n)^3D(x^4) &= (4m^3 + 3m^2n + mn^2)D(x)x^3 \\
 &\quad + (7m^2n + mn^2)xD(x)x^2 + (7mn^2 + m^2n)x^2D(x)x \\
 &\quad + (4n^3 + 3mn^2 + m^2n)x^3D(x), \quad x \in R.
 \end{aligned}$$

Putting $(m+n)x^2$ for x in (1) we obtain

$$\begin{aligned}
 (m+n)^3D(x^4) &= 2m(m+n)^2D(x^2)x^2 + 2n(m+n)^2x^2D(x^2) \\
 &= 2m(m+n)(2mD(x)x + 2nxD(x))x^2 \\
 &\quad + 2n(m+n)x^2(2mD(x)x + 2nxD(x)) \\
 &= 4m^2(m+n)D(x)x^3 + 4mn(m+n)xD(x)x^2 \\
 &\quad + 4mn(m+n)x^2D(x)x + 4n^2(m+n)x^3D(x), \quad x \in R.
 \end{aligned}$$

We have therefore

$$\begin{aligned}
 (6) \quad (m+n)^3D(x^4) &= 4m^2(m+n)D(x)x^3 + 4mn(m+n)xD(x)x^2 \\
 &\quad + 4mn(m+n)x^2D(x)x \\
 &\quad + 4n^2(m+n)x^3D(x), \quad x \in R.
 \end{aligned}$$

By comparing (5) and (6) we obtain

$$\begin{aligned}
 mn(n-m)D(x)x^3 + 3mn(m-n)xD(x)x^2 \\
 + 3mn(n-m)x^2D(x)x + mn(m-n)x^3D(x) = 0, \quad x \in R,
 \end{aligned}$$

whence it follows $D(x)x^3 - 3xD(x)x^2 + 3x^2D(x)x - x^3D(x) = 0$, $x \in R$, which can be written in the form $[D(x), x]_3 = 0$, $x \in R$. According to Theorem 3 it follows that $[D(x), x] = 0$, for all $x \in R$, which makes it possible to replace in (1) $D(x)x$ with $xD(x)$. We have therefore $(m+n)D(x^2) = 2(m+n)xD(x)$, $x \in R$, which reduces to $D(x^2) = 2xD(x)$, $x \in R$. Ap-

plying again the fact that D is commuting on R , we arrive at $D(x^2) = D(x)x + xD(x)$, $x \in R$. In other words, D is a Jordan derivation, whence it follows that D is a derivation by Herstein's result. Since by Posner's second theorem the existence of commuting nonzero derivation on a prime ring forces the ring to be commutative, one can conclude that the proof of the theorem is complete.

In our forthcoming paper [12] we prove the following result. Let R be a 2-torsion free semiprime ring and let $D : R \rightarrow R$ be a left Jordan derivation. In this case D is a derivation which maps R into $Z(R)$.

The result we have just mentioned above and Theorem 2 lead to the following conjecture.

CONJECTURE 1. *Let $m \geq 0$, $n \geq 0$ be fixed integers with $m + n \neq 0$, $m \neq n$ and let $D : R \rightarrow R$ be a (m, n) -Jordan derivation, where R is a semiprime ring with suitable torsion restrictions. In this case D is a derivation which maps R into $Z(R)$.*

An additive mapping $D : R \rightarrow R$, where R is an arbitrary ring, is called a Jordan triple derivation in case

$$(7) \quad D(xy x) = D(x)yx + xD(y)x + xyD(x)$$

is fulfilled for all pairs $x, y \in R$. One can easily prove that any Jordan derivation which maps a 2-torsion free ring into itself, is a Jordan triple derivation. Brešar [4] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. According to all these observations and Proposition 1 we continue with the definition and the conjecture below.

DEFINITION 2. Let $m \geq 0$, $n \geq 0$ with $m + n \neq 0$ be some fixed integers. An additive mapping $D : R \rightarrow R$, where R is an arbitrary ring, is called an (m, n) -Jordan triple derivation in case

$$(8) \quad \begin{aligned} (m+n)^2 D(xy x) = & m(n-m)D(x)xy + m(m-n)D(y)x^2 \\ & + n(n-m)x^2 D(y) + n(m-n)yx D(x) \\ & + m(3m+n)D(x)yx + 4mnx D(y)x \\ & + n(3n+m)xy D(x) \end{aligned}$$

holds for all pairs $x, y \in R$.

According to Proposition 1 any (m, n) -Jordan derivation on arbitrary 2-torsion free ring is an (m, n) -Jordan triple derivation. We conclude with the following conjecture.

CONJECTURE 2. *Let $m \geq 0$, $n \geq 0$ be some fixed integers with $m + n \neq 0$, $m \neq n$, and let $D : R \rightarrow R$ be an (m, n) -Jordan triple derivation, where R is a semiprime ring with suitable torsion restrictions. In this case D is a derivation which maps R into $Z(R)$.*

References

- [1] M. Brešar, J. Vukman, *Jordan derivations on prime rings*, Bull. Austral. Math. Soc. 37 (1988), 321–322.
- [2] M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. 104 (1988), 1003–1006.
- [3] M. Brešar, J. Vukman, *On left derivations and related mappings*, Proc. Amer. Math. Soc. 110 (1990), 7–16.
- [4] M. Brešar, *Jordan mappings of semiprime rings*, J. Algebra 127 (1989), 218–228.
- [5] M. Brešar, *Applying the theorem of functional identities*, Nova J. Math. Game Theory, and Algebra, 4 (1995), N 1, 43–54.
- [6] M. Brešar, B. Hvala, *On additive maps of prime rings, II*, Publ. Math. Debrecen 54/1-2 (1999), 39–54.
- [7] J. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. 53 (1975), 321–324.
- [8] Q. Deng, *On Jordan left derivations*, Math. J. Okayama Univ. 34 (1992), 145–147.
- [9] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1104–1110.
- [10] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., 8 (1957), 1093–1100.
- [11] J. Vukman, I. Kosi-Ulbl, *On some equations related to derivations in rings*, Internat. J. Math.&Math. Sci. 17 (2005), 2703–2710.
- [12] J. Vukman, *On left Jordan derivations of rings and Banach algebras*, Aequationes Math., to appear.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
 UNIVERSITY OF MARIBOR
 FNM, KOROŠKA 160
 2000 MARIBOR, SLOVENIA
 E-mail address: joso.vukman@uni-mb.si

Received January 29, 2008; revised version March 19, 2008.